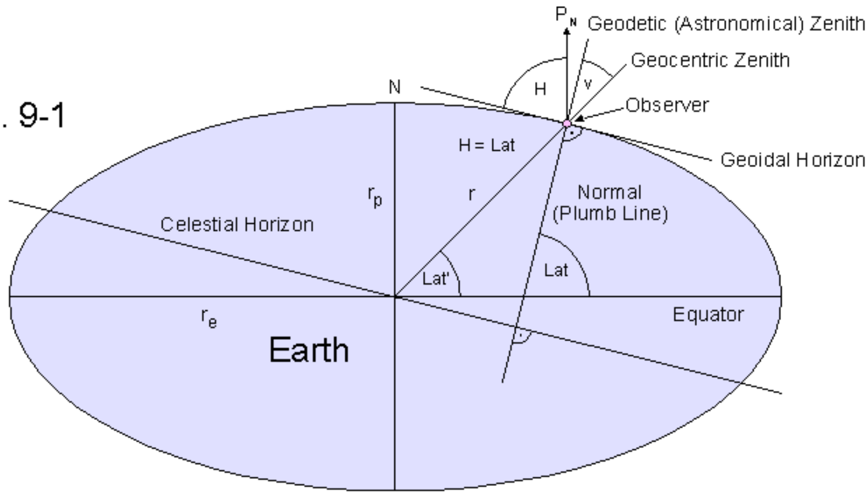


Geodetic Aspects of Celestial Navigation

The Ellipsoid

Celestial navigation is based upon the assumption that the earth is a sphere and, consequently, on the laws of spherical trigonometry. In reality, the shape of the earth is rather irregular and approximates an **oblate spheroid (ellipsoid)** resulting from two forces, **gravitation** and **centrifugal force**, acting on the viscous body of the earth. While gravitation alone would force the earth to assume the shape of a sphere, the state of lowest potential energy, the centrifugal force caused by the earth's rotation contracts the earth along the axis of rotation (polar axis) and stretches it along the plane of the equator. The local vector sum of both forces is called **gravity**.

A number of reference ellipsoids are in use to describe the shape of the earth, for example the **World Geodetic System (WGS)** ellipsoid of 1984. **The following considerations refer to the ellipsoid model of the earth which is sufficient for most navigational purposes**. *Fig.9-1* shows a meridional section of the ellipsoid.



Earth data (WGS 84 ellipsoid) :

Equatorial radius	r_e	6378137.0 m
Polar radius	r_p	6356752.3142 m
Flattening	$f = (r_e - r_p) / r_e$	1/298.25722

Due to the flattening of the earth, we have to distinguish between **geodetic** and **geocentric** latitude which would be the same if the earth were a sphere. The geodetic latitude of a given position, Lat, is the angle formed by the local normal (perpendicular) to the surface of the reference ellipsoid and the plane of the equator. The geocentric latitude, Lat', is the angle formed by the local radius vector and the plane of the equator. Geodetic and geocentric latitude are interrelated as follows:

$$\tan Lat' = (1 - f)^2 \cdot \tan Lat$$

Geodetic and geocentric latitude are equal at the poles and on the equator. At all other places, the geocentric latitude, Lat', is smaller than the geodetic latitude, Lat. As with the spherical earth model, geodetic and geocentric longitude are equal. Maps are always based upon **geodetic coordinates.** These are also referred to as **geographic coordinates**.

In the following, we will discuss the effects of the oblateness (flattening) of the earth on celestial navigation. Zenith distances (and altitudes) measured by the navigator always refer to the local plumb line which aligns itself with gravity and points to the **astronomical zenith**. Even the visible sea horizon correlates with the astronomical zenith since the water surface is perpendicular to the local plumb line.

With a homogeneous mass distribution throughout the ellipsoid, the plumb line coincides with the local normal to the ellipsoid which points to the **geodetic zenith**. Thus, astronomical and geodetic zenith are identical in this case.

The **geocentric zenith** is defined as the point where the extended local radius vector of the earth intersects the celestial sphere. The angular distance of the geodetic zenith from the **g**eocentric zenith is called **angle of the vertical**, **v**. The angle of the vertical is a function of the geodetic latitude. The following formula was proposed by *Smart* [9]:

$$v[''] \approx 692.666 \cdot \sin(2 \cdot Lat) - 1.163 \cdot \sin(4 \cdot Lat) + 0.026 \cdot \sin(6 \cdot Lat)$$

The coefficients of the above formula refer to the proportions of the WGS 84 ellipsoid.

The angle of the vertical at a given position equals the difference between geodetic and geocentric latitude (*Fig. 9-1*):

$$v = Lat - Lat'$$

The maximum value of v, occurring at 45° geographic latitude, is approx. 11.5'. Thus, the geocentric latitude of an observer being at 45°geodetic latitude is only 44° 48.5'. This difference is not negligible. Therefore, the navigator has to know if the coordinates of a fix obtained by astronomical observations are geodetic or geocentric. Altitudes are measured with respect to the sea horizon or an artificial horizon. Both correlate with the local plumb line which points to the geodetic (astronomical) zenith. Thus, the latter is the only reference available to the navigator. As demonstrated in *Fig. 9-1*, the altitude of the celestial north pole, P_N, (corrected altitude of Polaris) with respect to the geoidal horizon equals the geodetic, not the geocentric latitude. A noon latitude, being the sum or difference of the (geocentric) declination and the zenith distance with respect to the geodetic zenith would give the same result.

Assuming a homogeneous mass distribution within the (ellipsoidal) earth, latitudes obtained by celestial observations are geodetic latitudes since the navigator measures altitudes with respect to the local geodetic zenith (directly or indirectly).

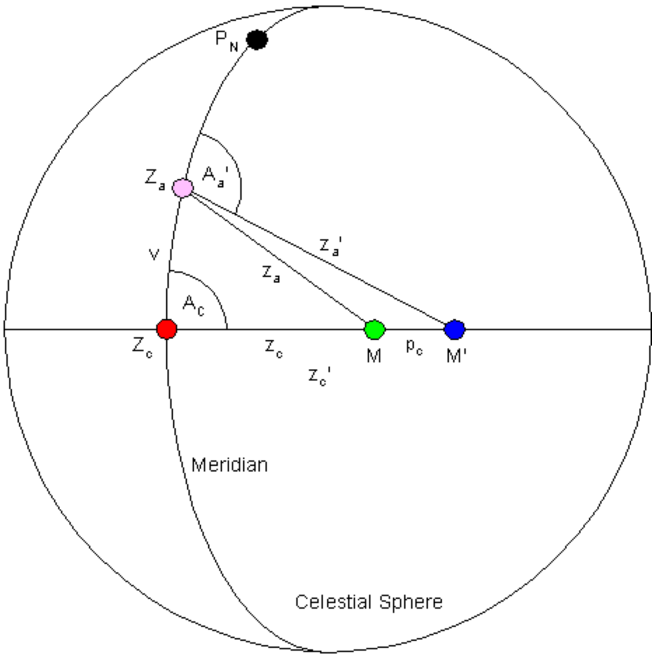
It is further important to know if the oblateness of the earth causes significant errors due to the fact that calculations of celestial navigation are based on a spherical earth model. According to the above values for polar radius and equatorial radius of the earth, the great circle distance of 1' is 1.849 km at the poles and 1.855 km at the equator. This small difference does not produce a significant error when plotting lines of position. It is therefore sufficient to use the adopted mean value (1 nautical mile = 1.852 km). However, when calculating the great circle distance (see chapter 11) of two locations thousands of nautical miles apart, the error caused by the oblateness of the earth can increase to several nautical miles. If extraordinary precision is required, the formulas for geodetic distance given in [2] should be used. A g**eodesic line** is the shortest path between two points on the surface of an ellipsoid. On the surface of a sphere, a geodesic line is the arc of a great circle.

The Parallax of the Moon

Due to the oblateness of the earth, the distance between geoidal and celestial horizon is not constant but can assume any value between r_p and r_e, depending on the observer's latitude. This has a measurable effect on the parallax of the moon since tabulated values for HP refer to the equatorial radius, r_e. The parallax of the moon is further affected by the displacement of the plumb line from the earth's center. A correction formula compensating both effects is given in chapter 2. The asymmetry of the plumb line with respect to the earth's center even causes a small (negligible) parallax in azimuth unless the moon is on the local meridian. In the following, we will calculate the effects of the oblateness of the earth on the parallax of the moon with the exact formulas of spherical astronomy [9]. For practical navigation, the simplified correction formulas given in chapter 2 are accurate enough.

Fig. 9-2 shows a projection of the astronomical zenith, Z_a, the geocentric zenith, Z_c, and the geographic position of the moon, M, on the **celestial sphere**, an imaginary hollow sphere of infinite diameter with the earth at its center.

Fig. 9-2



The geocentric zenith, Z_c, is the point where a straight line from the earth's center through the observer's position intersects the celestial sphere. The astronomical zenith, Z_a, is the point at which the plumb line going through the observer's position intersects the celestial sphere. Z_a and Z_c are on the same meridian. M is the projected geocentric position of the moon defined by Greenwich hour angle and declination. Unfortunately, the position of a body defined by GHA and Dec is commonly called geographic position (see chapter 3) although GHA and Dec are geocentric coordinates. M' is the point where a straight line from the observer through the moon's center intersects the celestial sphere. Z_c, M, and M' are on a great circle. The zenith distance measured by the observer is z_a' because the astronomical zenith is the available reference. The quantity we want to know is z_a, the astronomical zenith distance corrected for parallax in altitude. This is the angular distance of the moon from the astronomical zenith, measured by a fictitious observer at the earth's center.

The known quantities are v, A_a', and z_a'. In contrast to the astronomer, the navigator is usually not able to measure A_a' precisely. For navigational purposes, the calculated azimuth (see chapter 4) may be substituted for A_a'.

We have three spherical triangles, Z_aZ_cM', Z_aZ_cM, and Z_aMM'. First, we calculate z_c' from z_a', v, and A_a' using the **law of cosines for sides** (see chapter 10):

$$\cos z_c' = \cos z_a' \cdot \cos v + \sin z_a' \cdot \sin v \cdot \cos\left(180° - A_a'\right)$$

$$z_c' = \arccos\left(\cos z_a' \cdot \cos v - \sin z_a' \cdot \sin v \cdot \cos A_a'\right)$$

To obtain z_c, we first have to calculate the relative length ($r_e = 1$) of the radius vector, r, and the geocentric parallax, p_c:

$$p_c = \arcsin\left(\rho \cdot \sin HP \cdot \sin z_c'\right)$$

$$\rho = \frac{r}{r_e} = \sqrt{\frac{1-\left(2e^2 - e^4\right)\cdot \sin^2 Lat}{1 - e^2 \cdot \sin^2 Lat}} \qquad e^2 = 1 - \frac{r_p^2}{r_e^2}$$

HP is the equatorial horizontal parallax.

The geocentric zenith distance corrected for parallax is:

$$z_c = z'_c - p_c$$

Using the cosine formula again, we calculate A_c, the azimuth angle of the moon with respect to the geocentric zenith:

$$A_c = \arccos \frac{\cos z'_a - \cos z'_c \cdot \cos v}{\sin z'_c \cdot \sin v}$$

The astronomical zenith distance corrected for parallax is:

$$z_a = \arccos\left(\cos z_c \cdot \cos v + \sin z_c \cdot \sin v \cdot \cos A_c\right)$$

Thus, the **parallax in altitude** (astronomical) is:

$$PA = z'_a - z_a$$

For celestial navigation, the exact formulas of spherical astronomy are not needed, and the correction formula given in chapter 2 is accurate enough.

The small angle between M and M', measured at Z_a, is the **parallax in azimuth**, p_{az}:

$$p_{az} = \arccos \frac{\cos p_c - \cos z_a \cdot \cos z'_a}{\sin z_a \cdot \sin z'_a}$$

The correction for p_{az} is always applied so as to increase the angle formed by the azimuth line and the local meridian. For example, if Az_N is 315°, p_{az} is subtracted, and p_{az} is added if Az_N is 225°.

There is a simple formula for calculating the approximate parallax in azimuth:

$$p_{az} \approx f \cdot HP \cdot \frac{\sin(2 \cdot Lat) \cdot \sin Az_N}{\cos Hc}$$

This formula always returns the correct sign for p_{az}, and p_{az} is simply added to Az_N.

The parallax in azimuth does not exist when the moon is on the local meridian and is greatest when the moon ist east or west of the observer. It is further greatest at medium latitudes (45°) and non-existant when the observer is at one of the poles or on the equator (v = 0). Usually, the parallax in azimuth is only a fraction of an arcminute and therefore insignificant to celestial navigation. The parallax in azimuth increases with decreasing zenith distance.

Other celestial bodies do not require a correction for the oblateness of the earth since their parallaxes are very small compared with the parallax of the moon.

The Geoid

The earth is not **exactly** an oblate ellipsoid. The shape of the earth is more accurately described by the **geoid**, an equipotential surface of gravity.

The geoid has elevations and depressions caused by geographic features and a non-uniform mass distribution (materials of different density). Elevations occur at local accumulations of matter (mountains, ore deposits), depressions at local deficiencies of matter (valleys, lakes, caverns). The elevation or depression of each point of the geoid with respect to the reference ellipsoid is found by gravity measurement.

On the slope of an elevation or depression of the geoid, the plumb line (the normal to the geoid) does not coincide with the normal to the reference ellipsoid, and the astronomical zenith differs from the geodetic zenith. Thus, an **astronomical position** (obtained through uncorrected astronomical observations) may slightly differ from the geodetic position. The small angle formed by the local plumb line and the local normal to the reference ellipsoid is called **deflection of the vertical**. Usually, this angle is smaller than one arcminute, but greater deflections of the vertical have been reported, for example, in coastal waters with adjacent high mountains.

The local deflection of the vertical can be expressed in a latitude component and a longitude component. A position found by astronomical observations has to be corrected for both quantities to obtain the geodetic (geographic) position. The position error caused by the local deflection of the vertical is usually not relevant to celestial navigation at sea but is important to surveying and map-making where a much higher degree of accuracy is required.